Math 250 3.8 Implicit Differentiation

Objectives

1) Distinguish between implicit and explicit forms of equations
2) Use implicit differentiation to find the derivative of an implicit expression

Procedure for Implicit Differentiation

Given an equation containing x and y.

Step 1: Assume that y is a differentiable function of x, namely that $\frac{dy}{dx}$ exists and represents the rate of change of y with respect to x, the derivative we seek.

Step 2: Differentiate everything on both sides of the equation.

Use the chain rule when taking derivatives of y, namely, multiply by $\frac{dy}{dx}$.

Use the product, quotient and chain rules when any combination of x and y indicate.

Step 3: Isolate $\frac{dy}{dx}$

If $\frac{dy}{dx}$ appears only once, isolate it.

If $\frac{dy}{dx}$ appears twice or more, collect $\frac{dy}{dx}$ terms on the same side and non-$\frac{dy}{dx}$ terms on the other side; factor out $\frac{dy}{dx}$, then isolate by dividing.

Examples and Practice

1) Find the slope of the tangent line to the unit circle $x^2+y^2=1$ using implicit differentiation at

a. $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

b. $\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

c. $(1,0)$

d. $(0,1)$

e. $(-1,0)$

f. $(0,-1)$

2) Find $\frac{dy}{dx}$ by implicit differentiation for $x^2y+y^2x=-2$

3) Find the equation of the tangent line to $x^{2/3}+y^{2/3}=5$ at $(8,1)$

4) Find y'' if $x^2-y^2=36$

5) Find y'' when $\sin x+x^2y=10$

6) Find y'' in terms of x and y when $x^2y^2-2x=3$

7) Find $\frac{dy}{dx}$ when $(x+y)^3=x^3+y^3$

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Consider the unit circle, radius 1, center (0,0)

equation $(x-0)^2 + (y-0)^2 = 1^2$

$$x^2 + y^2 = 1$$

The equation $x^2 - y^2 = 1$ is an <u>implicit</u> form

because
- x and y both appear on one side of equation
- is not written in $f(x) =$ stuff w/ x form

This equation can be written in <u>explicit</u> form if we solve for y.

$$x^2 + y^2 = 1$$

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1-x^2}$$

but it requires two equations.

$$\text{and} \begin{cases} y = \sqrt{1-x^2} \\ y = -\sqrt{1-x^2} \end{cases}$$

Sometimes an implicit relationship can be solved for y, and other times it cannot.

<u>Implicit differentiation</u> is the process used to find $\frac{dy}{dx}$ from an implicit equation.

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① Find the slope of the tangent line to the unit circle $x^2+y^2=1$ using implicit differentiation.

at a) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

b) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

c) $(1,0)$

d) $(0,1)$

e) $(-1,0)$

f) $(0,-1)$

Differentiate all terms with respect to x.

Assume $y(x)$ so $\frac{dy}{dx}$ results by chain rule from all derivatives of y

and any $x \cdot y$ or $\frac{x}{y}$ must be product or quotient.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

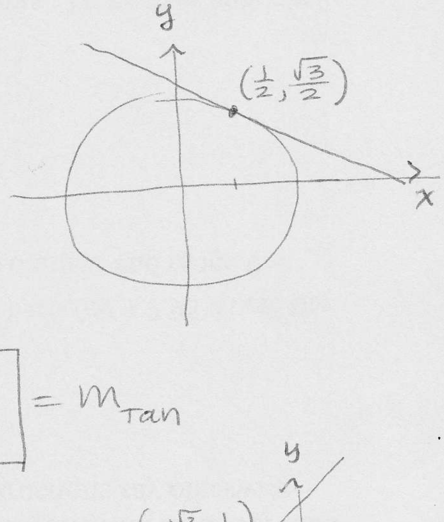
$$2x + 2y\frac{dy}{dx} = 0$$

Isolate $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x$$

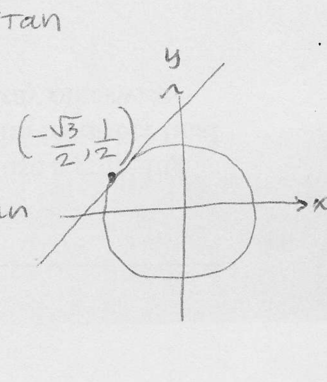
$$\frac{dy}{dx} = \frac{-2x}{+2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$



a) Evaluate $\left.\frac{dy}{dx}\right|_{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)} = \frac{-\left(\frac{1}{2}\right)}{\sqrt{3}/2} = -\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \boxed{\frac{-1}{\sqrt{3}}} = m_{Tan}$

b) Evaluate $\left.\frac{dy}{dx}\right|_{\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)} = \frac{-\left(\frac{-\sqrt{3}}{2}\right)}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \boxed{\sqrt{3}} = m_{Tan}$



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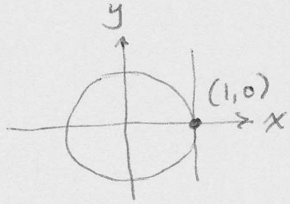
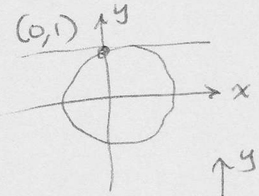
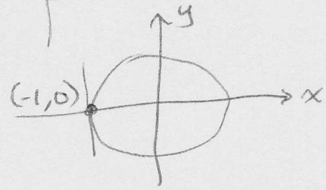
c) Evaluate $\left.\frac{dy}{dx}\right|_{(1,0)} = \frac{-1}{0} = \boxed{\text{undefined}}$



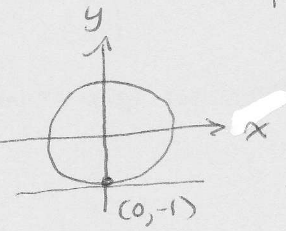
d) Evaluate $\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{-0}{1} = \boxed{0}$



e) Evaluate $\left.\frac{dy}{dx}\right|_{(-1,0)} = \frac{-(-1)}{0} = \boxed{\text{undefined}}$



f) Evaluate $\left.\frac{dy}{dx}\right|_{(0,-1)} = \frac{-0}{-1} = \boxed{0}$



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(2) Find $\frac{dy}{dx}$ by implicit differentiation

$$\underbrace{x^2y}_{\text{product rule}} + \underbrace{y^2x}_{\text{product rule}} = -2$$

$$\underbrace{x^2 \cdot \frac{d}{dx}[y] + y \cdot \frac{d}{dx}[x^2]} + \underbrace{y^2 \cdot \frac{d}{dx}[x] + x \cdot \frac{d}{dx}[y^2]} = \frac{d}{dx}[-2]$$

$$x^2 \cdot \frac{dy}{dx} + y \cdot 2x + y^2 \cdot 1 + x \cdot 2y\frac{dy}{dx} = 0$$

$$x^2 \frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$$

$$(x^2 + 2xy)\frac{dy}{dx} = -2xy - y^2$$

$$\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$$

$$\boxed{\frac{dy}{dx} = \frac{-y(2x+y)}{x(x+2y)}}$$

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Examples.

$8^{2/3}+1^{2/3}=5$ ✓
$2^2+1^2=5$ ✓

③ Find equation of tangent line to
$x^{2/3}+y^{2/3}=5$ at $(8,1)$.

Yes

To write eqn of line: need slope + point.
Get slope from derivative.
Equation is implicit — use implicit diff.

$$\tfrac{2}{3}x^{-1/3}+\tfrac{2}{3}y^{-1/3}\frac{dy}{dx}=0.$$

$$\frac{2}{3\sqrt[3]{y}}\frac{dy}{dx}=\frac{-2}{3\sqrt[3]{x}}$$

$$\frac{dy}{dx}=\frac{-\sqrt[3]{y}}{\sqrt[3]{x}}=\frac{-y^{1/3}}{x^{1/3}}$$

Evaluate derivative at $(8,1)$:

$$\left.\frac{dy}{dx}\right|_{(8,1)}=-\frac{\sqrt[3]{1}}{\sqrt[3]{8}}=\frac{-1}{2}$$ = slope of tangent line

Write equation — pt-slope form:

$$y-y_1=m(x-x_1)$$

$$y-1=-\frac{1}{2}(x-8)$$

Simplify

$$y=-\frac{1}{2}x+4+1$$

$$\boxed{y=-\frac{1}{2}x+5}$$

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(4) Find y'' if $x^2 - y^2 = 36$.

first derivative:

Yes

$$2x - 2y\frac{dy}{dx} = 0$$

$$-2y\frac{dy}{dx} = -2x$$

$$\frac{\cancel{d}y}{\cancel{d}x} = \frac{-2x}{-2y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Second derivative:

explicitly:

$$\frac{d^2y}{dx^2} = \frac{y \cdot 1 - x \cdot y'}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{y - x\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{y - \frac{x^2}{y}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$$

subst $x^2 - y^2 = 36$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-36}{y^3}}$$

use original equation to simplify

implicitly

$$y \cdot \frac{dy}{dx} = x$$

$$y \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{dy}{dx} = 1$$

$$y\frac{d^2y}{dx^2} + \left(\frac{x}{y}\right)^2 = 1$$

$$y\frac{d^2y}{dx^2} + \frac{x^2}{y^2} = 1$$

$$y\frac{d^2y}{dx^2} = 1 - \frac{x^2}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{1 - \frac{x^2}{y^2}}{y}$$

$$\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$$

subst $x^2 - y^2 = 36$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-36}{y^3}}$$

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(5) Find y'' when $\sin(x) + x^2 y = 10$

First derivative $\quad \underbrace{\cos(x) + \frac{d}{dx}(x^2) \cdot y + x^2 \cdot \frac{d}{dx}(y)}_{\text{product rule}} = 0$

option2 for y'' $\rightarrow \cos(x) + 2xy + x^2 \frac{dy}{dx} = 0$

$$x^2 \frac{dy}{dx} = -\cos(x) - 2xy$$

option1 for y'' $\rightarrow \frac{dy}{dx} = \frac{-\cos(x) - 2xy}{x^2}$

Second derivative

option 1: Quotient rule from $\frac{dy}{dx} = \frac{-\cos x - 2xy}{x^2}$

$$\frac{d^2y}{dx^2} = \frac{x^2 \cdot \frac{d}{dx}(-\cos x - 2xy) - (-\cos x - 2xy) \cdot \frac{d}{dx}(x^2)}{(x^2)^2}$$

product rule!

$$\frac{d^2y}{dx^2} = \frac{x^2 \cdot \left(+\cos x - 2\left(x \cdot \frac{dy}{dx} + y \cdot 1\right)\right) - 2x(-\cos x - 2xy)}{x^4}$$

$$= \frac{x^2\left(\cos x - 2x\frac{dy}{dx} - 2y\right) + 2x\cos x + 4x^2 y}{x^4}$$ dist.

$$= \frac{x^2 \cos x - 2x^3 \frac{dy}{dx} - 2x^2 y + 2x\cos x + 4x^2 y}{x^4}$$ dist

$$= \frac{x^2 \cos x - 2x^3 \frac{dy}{dx} + 2x\cos x + 2x^2 y}{x^4}$$ combine

$$= \frac{x\left(x\cos x - 2x^2 \frac{dy}{dx} + 2\cos x + 2xy\right)}{x^{4}}$$ factor x GCF

$$= \frac{x\cos x - 2x^2 \frac{dy}{dx} + 2\cos x + 2xy}{x^3}$$

cancel but... we still have $\frac{dy}{dx}$!

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replace $\frac{dy}{dx}$ by earlier result

$$= \frac{x\cos x - 2x^2\left[\frac{-\cos x - 2xy}{x^2}\right] + 2\cos x + 2xy}{x^3}$$

cancel x^2

$$= \frac{x\cos x - 2(-\cos x - 2xy) + 2\cos x + 2xy}{x^3}$$

$$= \frac{x\cos x + 2\cos x + 4xy + 2\cos x + 2xy}{x^3}$$ dist

$$\boxed{\frac{d^2y}{dx^2} = \frac{x\cos x + 4\cos x + 6xy}{x^3}}$$ combine

option 2: Implicit diff with product rule from

$$\cos x + 2xy + x^2\frac{dy}{dx} = 0$$

Differentiate $-\sin x + 2\underbrace{\left[\left(\frac{d}{dx}x\right)\cdot y + x\cdot\frac{dy}{dx}\right]}_{\text{product rule}} + \left[\left(\frac{d}{dx}x^2\right)\cdot\frac{dy}{dx} + x^2\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = 0$

"Take the derivative $\frac{d}{dx}$ of the first derivative $\frac{dy}{dx}$" = second derivative $\frac{d^2y}{dx^2}$

$$-\cos x + 2\left[y + x\frac{dy}{dx}\right] + \left[2x\frac{dy}{dx} + x^2\cdot\frac{d^2y}{dx^2}\right] = 0$$

$$-\cos x + 2y + 2x\frac{dy}{dx} + 2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0$$

$$-\cos x + 2y + 4x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0$$

replace $\frac{dy}{dx} = \frac{-\cos x - 2xy}{x^2}$

$$-\cos x + 2y + 4x\left(\frac{-\cos x - 2xy}{x^2}\right) + x^2\frac{d^2y}{dx^2} = 0$$

$$-\cos x + 2y - \frac{4\cos x}{x} - 8y + x^2\frac{d^2y}{dx^2} = 0$$

clear fraction mult all by x.

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$$-x\cos x + 2xy - 4\cos x - 8xy + x^3\frac{d^2y}{dx^2} = 0$$

combine like terms

$$-x\cos x - 4\cos x - 6xy + x^3\frac{d^2y}{dx^2} = 0$$

$$x^3\frac{d^2y}{dx^2} = x\cos x + 4\cos x + 6xy$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{x\cos x + 4\cos x + 6xy}{x^3}}$$

6 Method 1

Find $\frac{d^2y}{dx^2}$ in terms of x and y.

$$x^2y^2 - 2x = 3$$

Find first derivative

$$\underbrace{x^2 \cdot 2y\frac{dy}{dx} + y^2 \cdot 2x}_{\text{product rule on } x^2y^2} - 2 = 0$$

$$2x^2y\frac{dy}{dx} + 2xy^2 - 2 = 0 \qquad \text{tidy up}$$

$$2x^2y\frac{dy}{dx} = 2 - 2xy^2 \qquad \text{collect } \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2(1-xy^2)}{2x^2y} \qquad \text{isolate } \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1-xy^2}{x^2y} \qquad \text{factor and cancel 2}$$

Find second derivative

1) quotient rule
2) product rule for $\frac{d}{dx}$ top and again for $\frac{d}{dx}$ bottom.

$$\frac{d^2y}{dx^2} = \frac{(x^2y)\cdot(-1)\left[x\cdot 2y\frac{dy}{dx} + y^2\cdot 1\right] - (1-xy^2)\left[x^2\cdot 1\cdot\frac{dy}{dx} + y\cdot 2x\right]}{(x^2y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-2x^3y^2\frac{dy}{dx} - x^2y^3 - (1-xy^2)\left(x^2\frac{dy}{dx} + 2xy\right)}{x^4y^2} \qquad \text{dist tidy}$$

$$\frac{d^2y}{dx^2} = \frac{-2x^3y^2\frac{dy}{dx} - x^2y^3 - \left\{x^2\frac{dy}{dx} + 2xy - x^3y^2\frac{dy}{dx} - 2x^2y^3\right\}}{x^4y^2} \qquad \text{FOIL}$$

$$\frac{d^2y}{dx^2} = \frac{-2x^3y^2\frac{dy}{dx} - x^2y^3 - x^2\frac{dy}{dx} - 2xy + x^3y^2\frac{dy}{dx} + 2x^2y^3}{x^4y^2} \qquad \text{dist neg}$$

$$\frac{d^2y}{dx^2} = \frac{-x^3y^2\frac{dy}{dx} + x^2y^3 - x^2\frac{dy}{dx} - 2xy}{x^4y^2} \qquad \text{combine}$$

$$\frac{d^2y}{dx^2} = \frac{x\left[-x^2y^2\frac{dy}{dx} + xy^3 - x\frac{dy}{dx} - 2y\right]}{x^4y^2} \qquad \text{factor out } x.$$

$$\frac{d^2y}{dx^2} = \frac{-x^2y^2\frac{dy}{dx} + xy^3 - x\frac{dy}{dx} - 2y}{x^3y^2} \qquad \text{cancel } x.$$

$$\frac{d^2y}{dx^2} = \frac{-x^2y^2\left(\frac{1-xy^2}{x^2y}\right) + xy^3 - x\left(\frac{1-xy^2}{x^2y}\right) - 2y}{x^3y^2} \qquad \text{subst } \frac{dy}{dx} = \frac{1-xy^2}{x^2y}$$

6 con't

$$\frac{d^2y}{dx^2} = \frac{-y(1-xy^2) + xy^3 - \left(\frac{1-xy^2}{xy}\right) - 2y}{x^3y^2}$$ simplify

$$\frac{d^2y}{dx^2} = \frac{-y + xy^3 + xy^3 - \frac{1}{xy} + y - 2y}{x^3y^2}$$ dist

$$\frac{d^2y}{dx^2} = \frac{-2y + 2xy^3 - \frac{1}{xy}}{x^3y^2}$$ combine

$$\frac{d^2y}{dx^2} = \frac{-2xy^2 + 2x^2y^4 - 1}{x^4y^3}$$ mult by $\frac{xy}{xy}$ to clear frac.

$$\boxed{\frac{d^2y}{dx^2} = \frac{-2xy^2 + 2x^2y^4 - 1}{x^4y^3}}$$

Method 2

(6) using implicit differentiation for both derivatives.

$x^2y^2 - 2x = 3$

first derivative

$$\underbrace{x^2 \cdot 2y\frac{dy}{dx} + y^2 \cdot 2x}_{\text{product rule on } x^2y^2} - 2 = 0$$

$$2x^2y\frac{dy}{dx} + 2xy^2 - 2 = 0$$

$$2x^2y\frac{dy}{dx} = 2 - 2xy^2$$

$$\frac{dy}{dx} = \frac{2(1-xy^2)}{2x^2y}$$

$$\frac{dy}{dx} = \frac{1-xy^2}{x^2y}$$

Find second derivative

replace $\frac{dy}{dx}$ by y'

mult by x^2y to remove quotient

$x^2 \cdot y \cdot y' = 1 - xy^2$ — 3-way product rule on LHS

$2x \cdot y \cdot y' + x^2 \cdot y' \cdot y' + x^2 \cdot y \cdot y'' = -[x \cdot 2yy' + y^2 \cdot 1]$ — product rule RHS

$2xyy' + x^2 \cdot (y')^2 + x^2y \cdot y'' = -2xyy' - y^2$ — dist neg

$4xyy' + x^2(y')^2 + x^2y \cdot y'' - y^2 = 0$ — collect all on LHS.

$$4xy\left(\frac{1-xy^2}{x^2y}\right) + x^2\left(\frac{1-xy^2}{x^2y}\right)^2 + x^2yy'' + y^2 = 0$$

$4\left(\frac{1-xy^2}{x}\right) + \frac{(1-xy^2)^2}{x^2y^2} + x^2yy'' + y^2 = 0$ — simplify

$4xy^2(1-xy^2) + (1-xy^2)^2 + x^4y^3y'' + x^2y^4 = 0$ — clear frac

$4xy^2 - 4x^2y^4 + 1 - 2xy^2 + x^2y^4 + x^4y^3y'' + x^2y^4 = 0$ — FOIL, dist

$2xy^2 - 2x^2y^4 + 1 + x^4y^3y'' = 0$ — combine

$x^4y^3y'' = 2x^2y^4 - 2xy^2 - 1$ — collect y''

$\boxed{y'' = \frac{2x^2y^4 - 2xy^2 - 1}{x^4y^3}}$ — isolate y''

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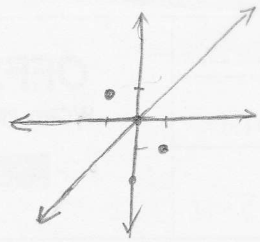
(7) $(x+y)^3 = x^3 + y^3$

chain rule (for $(x+y)^3$); chain rule for y (for y^3)



Graph is x-axis y-axis line y=x and two points (1,-1) and (-1,1)

$$3(x+y)^2 \cdot \left(1+\frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$$

Note: divide out 3 now or later

$$3(x+y)^2 \cdot \left(1+\frac{dy}{dx}\right) = 3\left(x^2+y^2\frac{dy}{dx}\right)$$

$$(x+y)^2 \cdot \left(1+\frac{dy}{dx}\right) = \left(x^2+y^2\frac{dy}{dx}\right)$$

$$(x^2+2xy+y^2)\left(1+\frac{dy}{dx}\right) = \left(x^2+y^2\frac{dy}{dx}\right)$$ FOIL

$$(x^2+2xy+y^2+x^2\frac{dy}{dx}+2xy\frac{dy}{dx}+y^2\frac{dy}{dx} = x^2+y^2\frac{dy}{dx}$$ more multiply

$$x^2\frac{dy}{dx}+2xy\frac{dy}{dx}+\cancel{y^2\frac{dy}{dx}}-\cancel{y^2\frac{dy}{dx}} = \cancel{x^2}-\cancel{x^2}-2xy-y^2$$ collect $\frac{dy}{dx}$

$$\frac{dy}{dx}(x^2+2xy) = -2xy-y^2$$ factor out $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-2xy-y^2}{x^2+2xy}$$ divide by coef of $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{-y(2x-y)}{x(x+2y)}}$$ factor.